Distribution of patches in tilings, tiling spaces and tiling dynamical systems (Natural extension of arithmetic algorithms and S-adic system)

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By

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Abstract

We introduce a basic theory of tilings, continuous hulls and tiling dynamical systems with detailed proofs. As a way to construct interesting tilings, tiling substitution is investigated. We put emphasis on two points: first, we introduce two topologies on the space of patches (and so the space of tilings) and describe the relation between them. Second, we discuss relations between properties of tilings and those of continuous hulls and tiling dynamical systems.

§ 1. Introduction

A cover of $\mathbb{R}^d$ by tiles such as polygons that overlap only on their borders is called a tiling. Decorations of walls in Islamic architecture suggest that the study of tilings is quite an old topic of mankind. In twentieth century scientific and popular interest arose after Penrose found a set of tiles (polygons) that can cover the plane $\mathbb{R}^2$ and form a tiling but only non-periodically. Here "non-periodic" means that translates of the tiling by non-zero vectors never coincide with the original tiling. Moreover, before Penrose, Berger [3] found a set of squares with colors on each edges such that, we can tile the plane $\mathbb{R}^2$ by such squares in a grid and in such a way that adjacent edges have the same color, but the resulting tilings are always non-periodic. This research is connected with a problem of logic raised by Wang [25].

A main feature of tilings such as tilings by Penrose (Penrose tilings) is their non-periodicity and almost-periodicity. [1] Here the term "non-periodic" has a clear meaning as described above but the word "almost periodic" is not clearly captured. However there are several signs that suggest certain non-periodic tilings are "close to" periodic ones and in such tilings tiles are

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[1] There is a similarity between such almost periodic tilings and almost periodic functions, which was first defined by Bohr and studied by many authors including Besicovitch, Bochner and von Neumann. (Consult textbooks such as [5], [4], [9], [8].) For example, we can find a similarity between the proof for compactness of the orbit closures of certain (FLC) tilings and the proof for compactness of the orbit closures of uniformly almost periodic functions. (Sometimes the compactness of the orbit closure is the definition of almost periodic function, but a proof for characterization of almost periodicity is similar.)

not distributed in a completely random way. For example, tilings such as Penrose tilings are repetitive, which means every finite pattern that appears in that tilings repeats in that tilings (see Definition 2.37). Moreover, for a certain tiling $\mathcal{T}$ of $\mathbb{R}^d$ its translates $\mathcal{T}+x$, $x\in\mathbb{R}^d$, come back arbitrary close to the original tiling $\mathcal{T}$ again and again ([7]).

This almost periodicity suggests that tilings can be used to study quasicrystals, which were first found by Shechtman [21]. Quasicrystals have long-range order but their diffraction patterns imply that they are not periodic. Such phenomena require a mathematical explanation and tilings can be used for such explanation. For example, for certain tilings, although they are not periodic, their diffraction measures are pure point. Here diffraction measure is a mathematical model of diffraction pattern and the above fact means that certain non-periodic tilings have long-range order. These days tilings are researched actively in connection to quasicrystals.

For the study of a tiling $\mathcal{T}$, its continuous hull (orbit closure, that is, the closure of the orbit $\{\mathcal{T}+x\mid x\in\mathbb{R}^d\}$) and its tiling dynamical system (an $\mathbb{R}^d$ action by translation on the continuous hull) are important. These objects are geometric analogues of subshifts and $\mathbb{N}$-actions on them in symbolic dynamics. On topology of continuous hulls, a book [19] is an excellent introduction. On tiling dynamical systems, see for example [23], [24], [11], [10], [18], [17] and a book [16]. On relations between tiling dynamical systems and diffraction measures, see for example [2], [20].

In this article we develop a basic theory of tilings, continuous hulls and tiling dynamical systems with detailed proofs. The argument is based on works by several authors such as Solomyak([23], [22], [24]), Lee-Solomyak([11], [10]) and Robinson ([18]).

In the theory we develop the following two points are stressed. First, we introduce two topologies on the space of all patches on $\mathbb{R}^d$: the cylinder topology (Definition 2.4) and the local matching topology (Definition 2.12). Tilings are Patches (Definition 2.2). Thus these two topologies define two topologies on a space of tilings. We investigate properties of these two topologies and relations between them. Often on the continuous hull of a tiling the relative topologies of these two coincide.

Second, relations between properties of tilings and those of continuous hulls and tiling dynamical systems are stressed. For example, relations between FLC of tilings and compactness of continuous hulls (Corollary 2.35), and repetitivity of tilings and minimality of tiling dynamical systems (Proposition 2.43) are fundamental. We can prove that for tilings from certain substitutions the corresponding tiling dynamical systems are not mixing (Theorem 3.32), and this is derived from a property of distribution of patches in tilings (Remark 3). Conversely, a property on distribution on patches is derived from a property of tiling dynamical systems (Theorem 2.44).

This article is organized as follows. In the second section we start from the definition of tilings and introduce their continuous hulls and tiling dynamical systems, followed by an explanation of important concepts such as FLC and repetitivity. In the third section we introduce substitution rules. Properties such as non-periodicity and repetitivity of tilings such as Penrose tilings are proved by their self-similar structure. Such structure is induced by (tiling) substitution rules, which are geometric versions of word substitutions in symbolic dynamics (for word substitution, see a book [14]). We explain important properties of tilings from substitutions. We finish the article with an appendix which covers basic terminology of dynamical systems.

Notation 1.1. *In this article* $\mathbb{Z}_{>0} = \{1, 2, \dots\}$ *is the set of integers larger than* 0*. In a metric space* $B(x, r)$ *denotes the open ball of radius* r *with its center* x*. In a topological space, if* S *is a subset of the space,* $\overline{S}$ *denotes the closure and* S° *denotes the open kernel. For* $\mathbb{R}^d$*, its Euclidean norm is represented by* $\|\cdot\|$*. We regard* $\mathbb{R}^d$ *as a metric space with the metric defined by this norm. For* $S \subset \mathbb{R}^d$*, its diameter is defined by* $\operatorname{diam} S = \sup_{x,y \in S} \|x - y\|$*. For* $S \subset \mathbb{R}^d$*, set* $-S = \{-x \mid x \in S\}$*. For* $S_1, S_2 \subset \mathbb{R}^d$*, set* $S_1 + S_2 = \{x + y \mid x \in S_1 \text{ and } y \in S_2\}$ *and* $S_1 - S_2 = S_1 + (-S_2)$*. If* $\mathcal{A}$ *is a set of subsets of* $\mathbb{R}^d$ *and* $S \subset \mathbb{R}^d$*, set* $\mathcal{A} + S = \{T + x \mid T \in \mathcal{A} \text{ and } x \in S\}$*. If* X *is a set, then the symbol* 2^X *denotes the set of all subsets of* X *and* $\operatorname{card} X$ *denotes its cardinality.*

§ 2. General theory of tilings, continuous hulls and tiling dynamical systems

§ 2.1. Definition of tilings and their properties

Here we introduce patches, tilings and topological spaces consisting of patches. Such topological spaces often admit an $\mathbb{R}^d$ action.

Definition 2.1. For any $\mathcal{P} \subset 2^{\mathbb{R}^d}$, the set $\operatorname{supp} \mathcal{P}$ defined by

$$\operatorname{supp} \mathcal{P} = \overline{\bigcup_{T \in \mathcal{P}} T}$$

is called the support of the set $\mathcal{P}$.

The support is the closure of the area that elements $T \in \mathcal{P}$ cover.

Definition 2.2. We fix $d \in \mathbb{Z}_{>0}$.

- An open, bounded and nonempty subset of $\mathbb{R}^d$ is called a tile.
- A set $\mathcal{P}$ of tiles such that $S, T \in \mathcal{P}$ and $S \neq T$ imply $S \cap T = \emptyset$ is called a patch. A patch $\mathcal{P}$ is said to be bounded if $\operatorname{supp} \mathcal{P}$ is bounded.
- A patch $\mathcal{T}$ such that $\operatorname{supp} \mathcal{T} = \mathbb{R}^d$ is called a tiling.
- For a tiling $\mathcal{T}$ and a vector $x \in \mathbb{R}^d$, suppose there exists $T \in \mathcal{T}$ such that $T + x \in \mathcal{T}$. Then we call x a return vector for $\mathcal{T}$.

Remark. In the literature, tile is defined in various ways. Often tiles are defined as compact sets which are "simple". What the word simple means depends on the authors. For example, in [1] a subset of $\mathbb{R}^d$ which is homeomorphic to the closed unit ball of $\mathbb{R}^d$ is called a tile.

Here we put the simplicity assumption by defining tiles as open sets. This change is not essential and the theory we develop becomes almost the same.

Often we consider labels on tiles in order to distinguish two tiles that are as sets the same. For example, one can prove unique ergodicity of certain tiling dynamical systems from substitutions by considering labels. On the other hand, considering labels gives an additional complexity in notation. Here we avoid considering labels, and when they are necessary we find a way round by giving a "puncture" to each tile (i.e. remove one point from each tile). Two tiles that are originally the same are after this procedure different if they have different punctures (see Example 3.5).

Remark. If $\mathcal{P}$ is a patch, then the set $\mathcal{P}$ is at most countable.

Definition 2.3. $\mathrm{Patch}(\mathbb{R}^d)$ denotes the set of all patches in $\mathbb{R}^d$. $\mathrm{Tiling}(\mathbb{R}^d)$ denotes the set of all tilings in $\mathbb{R}^d$.

Next we introduce two topologies on $\mathrm{Patch}(\mathbb{R}^d)$.

Definition 2.4. For $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and a neighborhood U of 0 in $\mathbb{R}^d$, set

$$C(U, \mathcal{P}) = \{\mathcal{Q} \in \mathrm{Patch}(\mathbb{R}^d) \mid \text{there exists } x \in U \text{ such that } \mathcal{P} + x \subset \mathcal{Q}\}.$$

Such sets are called cylinder sets. The topology generated by

$$\{C(U, \mathcal{P}) \mid U\text{: open neighborhood of 0 in } \mathbb{R}^d,\ \mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)\text{:bounded}\} \tag{2.1}$$

is called the cylinder topology.

Remark. The subbasis (2.1) is in fact a basis. For if $n \in \mathbb{Z}_{>0}$, $U_1, U_2, \dots U_n$ are open neighborhoods of 0, $\mathcal{P}_1, \mathcal{P}_2, \dots \mathcal{P}_n \in \mathrm{Patch}(\mathbb{R}^d)$ are bounded and

$$\mathcal{Q} \in \bigcap_i C(U_i, \mathcal{P}_i),$$

then for each i there is $x_i \in U_i$ such that $\mathcal{P}_i + x_i \subset \mathcal{Q}$. Set $\mathcal{P} = \bigcup_i (P_i + x_i)$. Then $\mathcal{P}$ is a bounded patch and if we take an open neighborhood U of 0 in $\mathbb{R}^d$ small enough, then

$$\mathcal{Q} \in C(U, \mathcal{P}) \subset \bigcap_i C(U_i, \mathcal{P}_i).$$

Lemma 2.5. *If $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the set*

$$\{C(U, \mathcal{Q}) \mid U\text{: neighborhood of } 0 \text{ in } \mathbb{R}^d \text{ and } \mathcal{Q} \subset \mathcal{P}\text{: bounded}\}$$

forms a neighborhood basis for $\mathcal{P}$ with respect to the cylinder topology.

Proof. Suppose $\mathcal{P} \in C(U, \mathcal{P}')$ for some open neighborhood U of 0 and a bounded $\mathcal{P}' \in \mathrm{Patch}(\mathbb{R}^d)$. Then there is $x \in U$ such that $\mathcal{P}' + x \subset \mathcal{P}$. If a neighborhood V of 0 is small enough,

$$\mathcal{P} \in C(V^\circ, \mathcal{P}' + x) \subset C(V, \mathcal{P}' + x) \subset C(U, \mathcal{P}').$$

□

Lemma 2.6. *The group $\mathbb{R}^d$ acts on $\mathrm{Patch}(\mathbb{R}^d)$ by translation:*

$$\mathrm{Patch}(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mathcal{P}, x) \mapsto \mathcal{P} + x \in \mathrm{Patch}(\mathbb{R}^d). \tag{2.2}$$

Furthermore this map is jointly continuous with respect to the cylinder topology.

Proof. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Take also a neighborhood O with respect to the cylinder topology of $P + x$. To prove the continuity of the map at $(\mathcal{P}, x)$, we may assume O is of the form $O = C(U, \mathcal{P}_0)$ where U is an open neighborhood of 0 in $\mathbb{R}^d$, $\mathcal{P}_0$ is bounded and $\mathcal{P}_0 \subset \mathcal{P} + x$ (cf. Lemma 2.5). Take a neighborhood V of x and a neighborhood V' of 0 such that if $y \in V$ and $z \in V'$, then $y - x + z \in U$. If $y \in V$ and $\mathcal{Q} \in C(V', \mathcal{P}_0 - x)$ (cf. Lemma 2.5), then there is $z \in V'$ such that $\mathcal{P}_0 - x + z \subset \mathcal{Q}$. We obtain $\mathcal{P}_0 + y - x + z \subset \mathcal{Q} + y$ and $\mathcal{Q} + y \in C(U, \mathcal{P}_0)$. □

Remark. If $\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, then $\mathcal{T} + x \in \mathrm{Tiling}(\mathbb{R}^d)$.

Next we define a uniform structure on $\mathrm{Patch}(\mathbb{R}^d)$ and the second topology on it. For generality of uniform space, see [6].

Definition 2.7. For any subset $\mathcal{P} \subset 2^{\mathbb{R}^d}$ and any subset $S \subset \mathbb{R}^d$ set

$$\mathcal{P} \cap S = \{T \in \mathcal{P} \mid T \subset S\}.$$

The next lemma is easy to prove.

Lemma 2.8. *If $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$, then $(\mathcal{P} \cap S) + x = (\mathcal{P} + x) \cap (S + x)$. If moreover $S_1 \subset S_2 \subset \mathbb{R}^d$, then $(\mathcal{P} \cap S_2) \cap S_1 = \mathcal{P} \cap S_1$.*

Definition 2.9. For a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of 0 in $\mathbb{R}^d$, set

$$\mathcal{U}_{K,V} = \{(\mathcal{P}_1, \mathcal{P}_2) \in \mathrm{Patch}(\mathbb{R}^d) \times \mathrm{Patch}(\mathbb{R}^d) \mid \text{there exists } x \in V \text{ such that } \mathcal{P}_1 \cap K = (\mathcal{P}_2 + x) \cap K\}.$$

Remark. If $K_1 \subset K_2$ and $V_1 \supset V_2$, then by Lemma 2.8, $\mathcal{U}_{K_1,V_1} \supset \mathcal{U}_{K_2,V_2}$.

The definition of notations such as $K - V$ and $-V$ is given in Notation 1.1.

Lemma 2.10. *The set*

$$\{\mathcal{U}_{K,V} \mid K \subset \mathbb{R}^d\text{: compact and } V\text{: a compact neighborhood of } 0 \text{ in } \mathbb{R}^d\} \tag{2.3}$$

forms a fundamental system of entourages for $\mathrm{Patch}(\mathbb{R}^d)$.

Proof. (1) For any K, V and $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, $(\mathcal{P}, \mathcal{P}) \in \mathcal{U}_{K,V}$, i.e. $\{(\mathcal{P}, \mathcal{P}) \mid \mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)\} \subset \mathcal{U}_{K,V}$.

(2)For any K and V, if $(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{U}_{K-V,-V}$, there is an $x \in V$ such that

$$\mathcal{P}_1 \cap (K - V) = (\mathcal{P}_2 - x) \cap (K - V).$$

By Lemma 2.8,

$$(\mathcal{P}_1 + x) \cap (x + K - V) = \mathcal{P}_2 \cap (x + K - V)$$

and

$$(\mathcal{P}_1 + x) \cap K = \mathcal{P}_2 \cap K.$$

Thus $(\mathcal{P}_2, \mathcal{P}_1) \in \mathcal{U}_{K,V}$ and $\mathcal{U}_{K-V,-V} \subset \mathcal{U}_{K,V}^{-1}$.

(3)Take two compact $K_1, K_2 \subset \mathbb{R}^d$ and compact neighborhoods V_1, V_2 of 0. Then

$$\mathcal{U}_{K_1 \cup K_2, V_1 \cap V_2} \subset \mathcal{U}_{K_1,V_1} \cap \mathcal{U}_{K_2,V_2}.$$

(4)Take K and V arbitrarily. Set $K' = (K - V) \cup K$. Take a compact neighborhood V' of 0 such that $V' + V' \subset V$. Note that $V' \subset V$. If $(\mathcal{P}_1, \mathcal{P}_2), (\mathcal{P}_2, \mathcal{P}_3) \in \mathcal{U}_{K',V'}$, then there are $x, y \in V'$ such that

$$\mathcal{P}_1 \cap K' = (\mathcal{P}_2 + x) \cap K', \text{ and } \mathcal{P}_2 \cap K' = (\mathcal{P}_3 + y) \cap K'.$$

Since $K' + x \supset K - V + x \supset K$, by Lemma 2.8,

$$\begin{aligned}
(\mathcal{P}_3 + x + y) \cap K &= ((\mathcal{P}_3 + x + y) \cap (K' + x)) \cap K \\
&= (((\mathcal{P}_3 + y) \cap K') + x) \cap K \\
&= ((\mathcal{P}_2 \cap K') + x) \cap K \\
&= (\mathcal{P}_2 + x) \cap K \\
&= ((\mathcal{P}_2 + x) \cap K') \cap K \\
&= (\mathcal{P}_1 \cap K') \cap K \\
&= \mathcal{P}_1 \cap K.
\end{aligned}$$

Thus $(\mathcal{P}_1, \mathcal{P}_3) \in \mathcal{U}_{K,V}$ and we obtain $\mathcal{U}_{K',V'}^2 \subset \mathcal{U}_{K,V}$. □

Definition 2.11. Let $\mathfrak{U}$ denote the set of all entourages generated by (2.3) and the uniform space constructed in this way is represented by $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$.

Recall that a uniform structure on a set defines a topology on that set. In this context $\{\mathcal{U}_{K,V}(\mathcal{P}) \mid K\text{: a compact subset of } \mathbb{R}^d, V\text{: a compact neighborhood of } 0\}$ form a neighborhood basis for $\mathcal{P}$. Here $\mathcal{U}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Patch}(\mathbb{R}^d) \mid (\mathcal{P}, \mathcal{Q}) \in \mathcal{U}\}$ for each $\mathcal{U} \in \mathfrak{U}$ and $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$.

Definition 2.12. The topology on $\mathrm{Patch}(\mathbb{R}^d)$ defined by the uniform structure $\mathfrak{U}$ is called the local matching topology.

Lemma 2.13. *Take $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Patch}(\mathbb{R}^d)$. If $T \in \mathcal{P}_1$ and for any compact neighborhood V of $0 \in \mathbb{R}^d$ there is $x_V \in V$ such that $T + x_V \in \mathcal{P}_2$, then $T \in \mathcal{P}_2$.*

Proof. Take $x \in T$. There is a neighborhood U of $0 \in \mathbb{R}^d$ such that if $y \in U$, then $x+y \in T$. This implies that $(T + y) \cap T \neq \emptyset$. Take a neighborhood U_0 of 0 such that $U_0 - U_0 \subset U$. If V_1, V_2

are compact neighborhoods of 0 such that $V_1, V_2 \subset U_0$, then $(T + x_{V_1}) \cap (T + x_{V_2}) \neq \emptyset$ and so $T + x_{V_1} = T + x_{V_2}$ since $\mathcal{P}_2$ is a patch. Take any $V \subset U_0$, then $(T - x_V) \setminus T = \emptyset$. For otherwise we can take $x \in (T - x_V) \setminus T$ and if $V' \subset U_0$ is small enough we have $x - x_{V'} \in T - x_V = T - x_{V'}$, which is a contradiction. Similarly $T \setminus (T - x_V) = \emptyset$ and $T = T + x_V \in \mathcal{P}_2$. □

Lemma 2.14. *The local matching topology is Hausdorff.*

Proof. Take $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Patch}(\mathbb{R}^d)$ and suppose for any compact $K \subset \mathbb{R}^d$ and any compact neighborhood V of $0 \in \mathbb{R}^d$, we have $(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{U}_{K,V}$. Take $T \in \mathcal{P}_1$ arbitrarily. For any compact $K \subset \mathbb{R}^d$ such that $T \subset K$ and any compact neighborhood V of 0, there is $x_V \in V$ such that

$$\mathcal{P}_1 \cap K = (\mathcal{P}_2 + x_V) \cap K.$$

Then $T - x_V \in \mathcal{P}_2$ and by Lemma 2.13 we have $T \in \mathcal{P}_2$. This argument shows that $\mathcal{P}_1 \subset \mathcal{P}_2$. By the same way $\mathcal{P}_2 \subset \mathcal{P}_1$ and $\mathcal{P}_1 = \mathcal{P}_2$. □

Lemma 2.15. *With respect to the local matching topology, the action* (2.2) *is jointly continuous.*

Proof. Take $P_0 \in \mathrm{Patch}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ arbitrarily. Consider a neighborhood of $P_0 + x_0$. To prove continuity of the action at (P_0, x_0), we may assume this neighborhood is of the form $\mathcal{U}_{K,V}(P_0 + x_0)$, where $K \subset \mathbb{R}^d$ is compact and V is a compact neighborhood of 0. Take a compact neighborhood V' of 0 such that $V' + V' \subset V$. If $P \in \mathcal{U}_{K-x_0,V'}(P_0)$ and $x \in x_0 - V'$, then there is $y \in V'$ such that

$$P_0 \cap (K - x_0) = (P + y) \cap (K - x_0).$$

It follows that

$$(P_0 + x_0) \cap K = (P + x + x_0 - x + y) \cap K.$$

Here $x_0 - x + y \in V$ and so $P + x \in \mathcal{U}_{K,V}(P_0 + x_0)$. □

Lemma 2.16. *For countably many $\mathcal{P}_1, \mathcal{P}_2, \ldots \in \mathrm{Patch}(\mathbb{R}^d)$ and a subset $S \subset \mathbb{R}^d$,*

$$\Big(\bigcup_n \mathcal{P}_n\Big) \cap S = \bigcup_n (\mathcal{P}_n \cap S).$$

Proposition 2.17. *The uniform space $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$ is complete.*

Proof. Since the uniform space is Hausdorff and has a countable fundamental system of entourages, it is metrizable ([6], chapter IX, §2.4). It suffices to show that all Cauchy sequences converge.

Take a Cauchy sequence $(\mathcal{P}_n)$. Set $V_n = \overline{B(0, \frac{1}{2^n})}$ (the closure of an open ball). Take also $R_n > 0$ for $n = 1, 2, \cdots$ such that $R_{n-1} + \frac{1}{2^n} < R_n$ and $\lim R_n = \infty$. Set $K_n = \overline{B(0, R_n)}$ for each n.

Since it suffices to show that a subsequence of $(\mathcal{P}_n)$ converges, we may assume that $(\mathcal{P}_k, \mathcal{P}_l) \in \mathcal{U}_{K_n,V_n}$ for any $k, l \geq n$. For each n there is $x_n \in V_n$ such that

$$(\mathcal{P}_n + x_n) \cap K_n = \mathcal{P}_{n+1} \cap K_n.$$

For each n the sequence $(\sum_{k=n}^{m} x_k)_m$ is a Cauchy sequence and so converges to a $y_n \in \mathbb{R}^d$. We have an estimate $\|y_n\| \leqq \frac{1}{2^{n-1}}$ and an equation $y_{n+1} + x_n = y_n$ for each n.

For each n

$$\begin{aligned}(\mathcal{P}_n + y_n) \cap (K_n + y_{n+1}) &= ((\mathcal{P}_n + x_n) \cap K_n) + y_{n+1} \\ &= (\mathcal{P}_{n+1} \cap K_n) + y_{n+1} \\ &= (\mathcal{P}_{n+1} + y_{n+1}) \cap (K_n + y_{n+1}).\end{aligned}$$

Since $K_{n-1} - y_{n+1} \subset K_n$,

$$(\mathcal{P}_n + y_n) \cap K_{n-1} = (\mathcal{P}_{n+1} + y_{n+1}) \cap K_{n-1}. \tag{2.4}$$

Set $\mathcal{P} = \bigcup_{n>1} (\mathcal{P}_n + y_n) \cap K_{n-1}$. By (2.4), $\mathcal{P}$ is a patch.

If $m \geq n + 1$,

$$\begin{aligned}(\mathcal{P}_m + y_m) \cap K_n &= ((\mathcal{P}_m + y_m) \cap K_{m-1}) \cap K_n \\ &= ((\mathcal{P}_{m+1} + y_{m+1}) \cap K_{m-1}) \cap K_n \\ &= (\mathcal{P}_{m+1} + y_{m+1}) \cap K_n.\end{aligned}$$

By induction we have

$$(\mathcal{P}_m + y_m) \cap K_n = (\mathcal{P}_{n+1} + y_{n+1}) \cap K_n$$

for each $m \geq n + 1$. For each $n > 1$, by Lemma 2.16,

$$\begin{aligned}\mathcal{P} \cap K_n &= (\bigcup_{m \geq n+1} (\mathcal{P}_m + y_m) \cap K_{m-1}) \cap K_n \\ &= (\mathcal{P}_{n+1} + y_{n+1}) \cap K_n.\end{aligned}$$

In other words $\mathcal{P}_{n+1} \in \mathcal{U}_{K_n,V_n}(\mathcal{P})$ for each n and $\mathcal{P}_n \to \mathcal{P}$. □

Proposition 2.18. *The local matching topology is stronger than the cylinder topology.*

Proof. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. For any bounded patch $\mathcal{P}_0 \subset \mathcal{P}$ and an open neighborhood U_0 of 0 in $\mathbb{R}^d$ (cf. Lemma 2.5), take a compact neighborhood U of 0 such that $U \subset U_0$ and set $K = U + \operatorname{supp} \mathcal{P}_0$. If $\mathcal{Q} \in \mathcal{U}_{K,U}^{-1}(\mathcal{P})$, then there is $x \in U$ such that

$$\mathcal{Q} \cap K = (\mathcal{P} + x) \cap K.$$

Since $\operatorname{supp}(\mathcal{P}_0 + x) \subset K$, $\mathcal{P}_0 + x \subset (\mathcal{P} + x) \cap K \subset \mathcal{Q}$. Then $\mathcal{Q} \in C(U, \mathcal{P}_0) \subset C(U_0, \mathcal{P}_0)$. This argument shows that $\mathcal{U}_{K,U}^{-1}(\mathcal{P}) \subset C(U_0, \mathcal{P}_0)$. □

Lemma 2.19. *Suppose $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Patch}(\mathbb{R}^d)$ and $\mathcal{P}_1 \subset \mathcal{P}_2$. Take $S \subset \mathbb{R}^d$ such that $\operatorname{supp} \mathcal{P}_1 \supset S$. Then $\mathcal{P}_1 \cap S = \mathcal{P}_2 \cap S$.*

Definition 2.20. For each $R > 0$, set

$$\mathrm{Tiling}_R(\mathbb{R}^d) := \{\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d) \mid \sup_{T \in \mathcal{T}} \operatorname{diam} T < R\}.$$

Proposition 2.21. *For any $R > 0$, on $\mathrm{Tiling}_R(\mathbb{R}^d)$, the relative topologies of local matching topology and cylinder topology coincide.*

Proof. Take $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$. Take also a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. Set $K' = K + \overline{B(0, R)}$ and $\mathcal{P}_0 = \mathcal{T} \cap K'$. Note that $\operatorname{supp} \mathcal{P}_0 \supset K$. If $\mathcal{S} \in C(-V, \mathcal{P}_0) \cap \mathrm{Tiling}_R(\mathbb{R}^d)$, there is $x \in V$ such that $\mathcal{P}_0 - x \subset \mathcal{S}$. By Lemma 2.8 and Lemma 2.19,

$$(\mathcal{S} + x) \cap K = \mathcal{P}_0 \cap K = (\mathcal{T} \cap K') \cap K = \mathcal{T} \cap K,$$

and $\mathcal{S} \in \mathcal{U}_{K,V}(\mathcal{T})$. Hence

$$\mathcal{T} \in C(-V, \mathcal{P}_0) \cap \mathrm{Tiling}_R(\mathbb{R}^d) \subset \mathcal{U}_{K,V}(\mathcal{T}).$$

We see on $\mathrm{Tiling}_R(\mathbb{R}^d)$ the cylinder topology is stronger than the local matching topology and together with Proposition 2.18 we see they are equal on $\mathrm{Tiling}_R(\mathbb{R}^d)$. $\square$

Remark. With respect to the local matching topology, $\mathrm{Tiling}_R(\mathbb{R}^d)$ is a closed subset of $\mathrm{Patch}(\mathbb{R}^d)$. However $\mathrm{Tiling}(\mathbb{R}^d)$ is not closed in $\mathrm{Patch}(\mathbb{R}^d)$ as the following example shows.

Example 2.22. Consider a tiling $\mathcal{T}_s$ of $\mathbb{R}^d$ defined by $\mathcal{T}_s = \{(0, 1)^d + x \mid x \in \mathbb{Z}^d\}$. We start from this tiling $\mathcal{T}_s$ and replace tiles with larger ones. For any $n \in \mathbb{Z}_{>0}$, choose $x_n \in \mathbb{Z}^d$ such that for any two distinct n and m, $((0, n)^d + x_n) \cap ((0, m)^d + x_m) = \emptyset$. To $\mathcal{T}_s$, we add tiles $(0, n)^d + x_n, n = 2, 3, \ldots$ and remove tiles with side-length 1 that intersect these tiles with side-length $2, 3, \ldots$. The resulting tiling is represented by $\mathcal{T}_s'$ and this consists of translates of $(0, n)^d$, $n = 1, 2, 3, \ldots$. This tiling is not in $\mathrm{Tiling}_R(\mathbb{R}^d)$ for any $R > 0$ and a sequence $(\mathcal{T}_s' - x_n - y_n)_n$, where $y_n = (\frac{1}{2}n, \frac{1}{2}n, \ldots, \frac{1}{2}n)$ for each n, converges to $\emptyset$ with respect to the local matching topology.

§ 2.2. Finite local complexity and finite tile type

Definition 2.23. On $2^{\mathbb{R}^d}$ (the set of all subsets of $\mathbb{R}^d$), define an equivalence relation $\approx$ by

$$A \approx B \iff \text{there exists } x \in \mathbb{R}^d \text{ such that } A = B + x.$$

On the set $2^{2^{\mathbb{R}^d}}$ of all subsets of $2^{\mathbb{R}^d}$, we define an equivalence relation $\sim$ by

$$\mathcal{P}_1 \sim \mathcal{P}_2 \iff \text{there exists } x \in \mathbb{R}^d \text{ such that } \mathcal{P}_1 = \mathcal{P}_2 + x.$$

Definition 2.24. An element $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ has finite local complexity (FLC) if the quotient set

$$\{(\mathcal{P}+x) \cap K \mid x \in \mathbb{R}^d\}/\!\sim$$

is finite for any compact $K \subset \mathbb{R}^d$.

Definition 2.25. An element $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ has finite tile type (FTT) if $\mathcal{P}/\!\approx$ is finite. In this case there exists a finite set $\mathcal{A}$ of tiles such that

- For any $P \in \mathcal{A}$, we have $0 \in P$, and
- For any $T \in \mathcal{P}$, there is a unique $P \in \mathcal{A}$ and a (necessarily unique) $x \in \mathbb{R}^d$ such that $T = P + x$.

Such a set $\mathcal{A}$ is called an alphabet for the FTT patch $\mathcal{P}$.

Given a finite non-empty set $\mathcal{A}$ of tiles that are not pairwise translationally equivalent, for any $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$ set $c_{\mathcal{A}}(P+x) = x$. For $\mathcal{P} \subset \mathcal{A} + \mathbb{R}^d$, set $c_{\mathcal{A}}(\mathcal{P}) = \{c_{\mathcal{A}}(T) \mid T \in \mathcal{P}\}$.

In Proposition 2.28 we give a characterization of FLC and FTT.

Definition 2.26. For a patch $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and $S \subset \mathbb{R}^d$, set

$$\mathcal{P} \sqcap S = \{T \in \mathcal{P} \mid \overline{T} \cap S \neq \emptyset\}.$$

Lemma 2.27. *For any subsets $\Pi_1, \Pi_2 \subset 2^{2^{\mathbb{R}^d}}$, suppose the following conditions;*

- *for any $\mathcal{P}_1 \in \Pi_1$ there are $\mathcal{P}_2 \in \Pi_2$ and $x \in \mathbb{R}^d$ such that $\mathcal{P}_1 + x \subset \mathcal{P}_2$,*
- *each $\mathcal{P}_2 \in \Pi_2$ is finite, and*
- *$\Pi_2/\!\sim$ is finite.*

Then $\Pi_1/\!\sim$ is finite.

Proposition 2.28. *For $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the following conditions are equivalent;*

1. *$\mathcal{P}$ has FTT and FLC.*
2. *$\mathcal{P}$ has FTT and $\{\mathcal{P}' \subset \mathcal{P} \mid \operatorname{diam}\operatorname{supp}\mathcal{P}' < R\}/\!\sim$ is finite for all $R > 0$.*
3. *$\{\mathcal{P} \cap B(x,R) \mid x \in \mathbb{R}^d\}/\!\sim$ is finite for any $R > 0$ and $\mathcal{P}$ has FTT.*
4. *$\{\mathcal{P} \sqcap (K+x) \mid x \in \mathbb{R}^d\}/\!\sim$ is finite for any compact $K \subset \mathbb{R}^d$.*
5. *$\mathcal{P}$ has FTT and $c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})$ is discrete and closed in $\mathbb{R}^d$, for any alphabet $\mathcal{A}$.*

Proof. 1⇒2. For any $R > 0$, if $\mathcal{P}' \subset \mathcal{P}$ and $\operatorname{diam}\operatorname{supp}\mathcal{P}' < R$, either $\mathcal{P}' = \emptyset$ or we can take $x \in \operatorname{supp}\mathcal{P}'$. In the latter case $\mathcal{P}' \subset \mathcal{P} \cap (x + \overline{B(0,R)})$ and Lemma 2.27 applies.

2⇒3. For any $x \in \mathbb{R}^d$, we have $\operatorname{diam}\operatorname{supp}(\mathcal{P} \cap B(x,R)) < 2R$. Lemma 2.27 applies.

3⇒4. Set $r = \max_{T\in\mathcal{P}} \operatorname{diam} T$. For any compact $K \subset \mathbb{R}^d$, take $R > 0$ such that $K \subset B(0, R - r)$. For any $x \in \mathbb{R}^d$ we have $\mathcal{P} \sqcap (K + x) \subset \mathcal{P} \cap B(x, R)$ and Lemma 2.27 implies (4).

4⇒5.First by taking $K = \{0\}$ we see $\mathcal{P}/\approx$ is finite and so $\mathcal{P}$ has FTT. Take $R > 0$ arbitrarily. We shall show that $(c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})) \cap B(0, R)$ is finite. Set $K = \overline{B(0, R)}$. There is a finite $F \subset \mathbb{R}^d$ such that if $x \in \mathbb{R}^d$ there are $y \in F$ and $z \in \mathbb{R}^d$ for which

$$\mathcal{P} \sqcap (K + x) = (\mathcal{P} \sqcap (K + y)) + z.$$

Take $a \in (c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})) \cap B(0, R)$. Then there are $P_1, P_2 \in \mathcal{A}$ and $a_1, a_2 \in \mathbb{R}^d$ such that $P_i + a_i \in \mathcal{P}$ $(i = 1, 2)$ and $a = a_1 - a_2$. By $a_2 \in P_2 + a_2$ and $\|a_1 - a_2\| < R$, we have $P_2 + a_2 \in \mathcal{P} \sqcap (K + a_1)$, and

$$\begin{aligned} a \in &(c_{\mathcal{A}}(\mathcal{P} \sqcap (K + a_1)) - c_{\mathcal{A}}(\mathcal{P} \sqcap (K + a_1)) \\ = &(c_{\mathcal{A}}(\mathcal{P} \sqcap (K + b)) - c_{\mathcal{A}}(\mathcal{P} \sqcap (K + b))) \end{aligned}$$

for some $b \in F$. Hence

$$(c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})) \cap B(0, R) \subset \bigcup_{b\in F} (c_{\mathcal{A}}(\mathcal{P} \sqcap (K + b)) - c_{\mathcal{A}}(\mathcal{P} \sqcap (K + b))). \tag{2.5}$$

Since F is finite and $\mathcal{P} \sqcap (K + b)$ is finite by FTT, the right-hand side of (2.5) is finite.

5⇒1. Take a compact $K \subset \mathbb{R}^d$ arbitrarily. Set $C = (c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})) \cap (K - K)$ and $\mathcal{P}' = \mathcal{A} + C$. If $x \in \mathbb{R}^d$ and $\mathcal{P} \cap (K + x) \neq \emptyset$, then take $P_0 \in \mathcal{A}$ and $x_0 \in \mathbb{R}^d$ such that $P_0 + x_0 \in \mathcal{P} \cap (K + x)$. If we arbitrarily take $P_1 \in \mathcal{A}$ and $x_1 \in \mathbb{R}^d$ such that $P_1 + x_1 \in \mathcal{P} \cap (K + x)$, then $x_1 - x_0 \in C$. This implies that $\mathcal{P} \cap (K + x) - x_0 \subset \mathcal{P}'$. Since $\mathcal{P}'$ is finite, by Lemma 2.27 $\{\mathcal{P} \cap (K + x) \mid x \in \mathbb{R}^d\}/\sim$ is finite. □

Remark. Example 2.22 is an example of tiling which has FLC but does not have FTT.

Definition 2.29. For $\mathcal{P} \in \operatorname{Patch}(\mathbb{R}^d)$, set

$$X_{\mathcal{P}} = \overline{\{\mathcal{P} + x \mid x \in \mathbb{R}^d\}}$$

with respect to the local matching topology.

Lemma 2.30. *If $\mathcal{P} \in \operatorname{Patch}(\mathbb{R}^d)$, $\mathcal{Q} \in X_{\mathcal{P}}$ and $x \in \mathbb{R}^d$, then $\mathcal{Q} + x \in X_{\mathcal{P}}$.*

Proof. There is a sequence (x_n) of $\mathbb{R}^d$ such that $\mathcal{Q} = \lim_n P + x_n$. By Lemma 2.15 $\mathcal{Q} + x = \lim \mathcal{P} + x_n + x \in X_{\mathcal{P}}$. □

Definition 2.31. A subset $X \subset \operatorname{Patch}(\mathbb{R}^d)$ has FLC if

$$\{\mathcal{P} \cap (K + x) \mid x \in \mathbb{R}^d, \mathcal{P} \in X\}/\sim$$

is finite for any compact $K \subset \mathbb{R}^d$.

Remark. If X is invariant under translation, X has FLC if and only if

$$\{\mathcal{P} \cap K \mid \mathcal{P} \in X\}/\sim$$

is finite for any compact $K \subset \mathbb{R}^d$. If there are only finitely many tile types in X, that is, $(\bigcup_{\mathcal{P}\in X} \mathcal{P})/\approx$ is finite, then by Lemma 2.27 X has FLC if and only if

$$\{\mathcal{P} \cap B(x, R) \mid x \in \mathbb{R}^d, \mathcal{P} \in X\}/\sim$$

is finite for any $R > 0$.

Lemma 2.32. *Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. Then the following two conditions are equivalent;*

1. *$\mathcal{P}$ has FLC.*
2. *$X_{\mathcal{P}}$ has FLC.*

Proof. 1⇒2. Take any compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. If $\mathcal{Q} \in X_{\mathcal{P}}$, then there is $x \in \mathbb{R}^d$ such that $\mathcal{P} + x \in \mathcal{U}_{K,V}(\mathcal{Q})$. This implies that there is $y \in V$ such that $(\mathcal{P} + x + y) \cap K = \mathcal{Q} \cap K$ and so

$$\{\mathcal{Q} \cap K \mid \mathcal{Q} \in X_{\mathcal{P}}\} = \{(\mathcal{P} + x) \cap K \mid x \in \mathbb{R}^d\}.$$

Since $X_{\mathcal{P}}$ is translation invariant (Lemma 2.30), we see $X_{\mathcal{P}}$ has FLC.

2⇒1. This direction is clear because $\{\mathcal{P} + x \mid x \in \mathbb{R}^d\} \subset X_{\mathcal{P}}$. □

Lemma 2.33. *Let X be an FLC subspace of $\mathrm{Patch}(\mathbb{R}^d)$. For any sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of X, any compact $K \subset \mathbb{R}^d$ and any compact neighborhood V of $0 \in \mathbb{R}^d$, we can take a subsequence $\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, \ldots$ of $(\mathcal{P}_n)_n$ such that $(\mathcal{P}_{n_j}, \mathcal{P}_{n_k}) \in \mathcal{U}_{K,V}$ for any $j, k > 0$.*

Proof. Set $K' = K - V$. By FLC there is a subsequence $\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, \ldots$ and $x_1, x_2, \ldots \in \mathbb{R}^d$ such that for any $j > 0$ we have

$$\mathcal{P}_{n_1} \cap K' = (\mathcal{P}_{n_j} \cap K') + x_j.$$

If $\mathcal{P}_{n_1} \cap K' = \emptyset$, we have nothing to prove and we may assume that we can take $T \in \mathcal{P}_{n_1} \cap K'$. Take $x \in T$, then $x - x_j \in K'$ for each j and we see $(x_j)_j$ is a bounded sequence. By taking subsequence again we may assume that $x_j - x_k \in V$ for any j, k. For any $j, k > 0$,

$$\mathcal{P}_{n_k} \cap K' = (\mathcal{P}_{n_j} \cap K') + x_j - x_k = (\mathcal{P}_{n_j} + x_j - x_k) \cap (K' + x_j - x_k)$$

and by Lemma 2.8,

$$\mathcal{P}_{n_k} \cap K = (\mathcal{P}_{n_j} + x_j - x_k) \cap K,$$

which implies $(\mathcal{P}_{n_k}, \mathcal{P}_{n_j}) \in \mathcal{U}_{K,V}$. □

Note that since the uniform space $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$ is metrizable, for any $X \subset \mathrm{Patch}(\mathbb{R}^d)$ the following two conditions are equivalent:

- X is totally bounded, that is, for any $\mathcal{U} \in \mathfrak{U}$ there is a finite $F \subset X$ such that $X \subset \bigcup_{\mathcal{P} \in F} \mathcal{U}(\mathcal{P})$.
- For any sequence in X, there is a Cauchy subsequence of it.

Note also that any $X \subset \mathrm{Patch}(\mathbb{R}^d)$ is compact if and only if it is closed and totally bounded.

Lemma 2.34. *For any $X \subset \mathrm{Patch}(\mathbb{R}^d)$, consider the following conditions;*

1. *X has FLC.*
2. *X is totally bounded with respect to $\mathfrak{U}$.*

Then condition 1 always implies condition 2 and the converse holds if X is invariant under translation and the set $(\bigcup_{\mathcal{P} \in X} \mathcal{P})/\approx$ is finite (that is, there are only finitely many tile types up to translation).

Proof. 1⇒2. Take countably many open sets $O_1, O_2, \dots$ and a countable neighborhood basis $\{V_n \mid n > 0\}$ of 0 consisting of compact sets such that

- $K_n := \overline{O_n}$ is compact for each n, and
- $\bigcup_n O_n = \mathbb{R}^d$.

Take a sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of X. By Lemma 2.33, we can take a subsequence $(\mathcal{P}_n^{(1)})$ of $(\mathcal{P}_n)$ such that $(\mathcal{P}_n^{(1)}, \mathcal{P}_m^{(1)}) \in \mathcal{U}_{K_1, V_1}$ for any $n, m > 0$. We further take a subsequence $(\mathcal{P}_n^{(2)})$ of $(\mathcal{P}_n^{(1)})$ such that $(\mathcal{P}_n^{(2)}, \mathcal{P}_m^{(2)}) \in \mathcal{U}_{K_2, V_2}$ for any $n, m > 0$. Proceeding in this way we can take subsequences $(\mathcal{P}_n^{(k)})_n$ for $k = 1, 2, \dots$. Set $Q_n = \mathcal{P}_n^{(n)}$ for each n, then $(\mathcal{Q}_n)_n$ is a Cauchy subsequence of $(\mathcal{P}_n)_n$.

2⇒1. Assume X is invariant under translation and $\bigcup_{\mathcal{P} \in X} \mathcal{P}/\approx$ is finite. Take a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of 0. By condition 2 there is a finite set $F \subset X$ such that

$$X \subset \bigcup_{\mathcal{P} \in F} \mathcal{U}_{V+K,V}(\mathcal{P}).$$

For any $\mathcal{Q} \in X$ there are $\mathcal{P} \in F$ and $x \in V$ such that $(\mathcal{Q} + x) \cap (K + V) = \mathcal{P} \cap (K + V)$, and

$$(\mathcal{Q} \cap K) + x = (\mathcal{Q} + x) \cap (K + x) \subset (\mathcal{Q} + x) \cap (K + V) = \mathcal{P} \cap (K + V).$$

Since $\mathcal{P} \cap (K + V)$ is finite for each $\mathcal{P} \in F$, by Lemma 2.27 $\{\mathcal{Q} \cap K \mid \mathcal{Q} \in X\}/\sim$ is finite. □

Corollary 2.35. *Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. Consider the following two conditions;*

1. *$\mathcal{P}$ has FLC.*
2. *$X_{\mathcal{P}}$ is compact with respect to the local matching topology.*

Then 1 always implies 2 and if $\mathcal{P}$ has FTT 2 implies 1.

Proof. Clear by Lemma 2.32, Lemma 2.34 and the fact that if $\mathcal{P}$ has FTT then $(\bigcup_{\mathcal{Q} \in X_{\mathcal{P}}} \mathcal{Q})/\approx$ is finite. □

Remark. If a tiling $\mathcal{T}$ has FTT, then on $X_{\mathcal{T}}$ the cylinder topology and the local matching topology coincide (Proposition 2.21). Thus if a tiling $\mathcal{T}$ has FLC and FTT the space $X_{\mathcal{T}}$ is compact with respect to both topologies.

§ 2.3. Repetitivity

Definition 2.36. A subset $S \subset \mathbb{R}^d$ is said to be relatively dense if there is a compact $K \subset \mathbb{R}^d$ such that $S + K = \mathbb{R}^d$.

Definition 2.37. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. $\mathcal{P}$ is said to be repetitive if for any bounded patch $\mathcal{Q} \subset \mathcal{P}$, the set

$$\{x \in \mathbb{R}^d \mid \mathcal{Q} + x \subset \mathcal{P}\}$$

is relatively dense.

Lemma 2.38. *For any $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the following two conditions are equivalent;*

1. *$\mathcal{P}$ is repetitive.*
2. *For any bounded $\mathcal{Q} \subset \mathcal{P}$, there is an $R > 0$ such that the following condition holds:*

$$\text{For any } a \in \mathbb{R}^d \text{ there is } x \in \mathbb{R}^d \text{ such that } \mathcal{P} \cap B(a, R) \supset \mathcal{Q} + x.$$

Proof. 1⇒2. Take a bounded $\mathcal{Q} \subset \mathcal{P}$. We may assume $\mathcal{Q} \neq \emptyset$. Take a translate $\mathcal{Q}'$ of $\mathcal{Q}$ such that $0 \in \operatorname{supp} \mathcal{Q}'$. Since $S = \{x \in \mathbb{R}^d \mid \mathcal{Q}' + x \subset \mathcal{P}\}$ is relatively dense, there is $R_0 > 0$ such that $S + B(0, R_0) = \mathbb{R}^d$. For any $a \in \mathbb{R}^d$ there is $x \in S \cap B(a, R_0)$. Then $\mathcal{Q}' + x \subset \mathcal{P} \cap B(a, R_0 + \operatorname{diam} \operatorname{supp} \mathcal{Q})$. Thus 2 is satisfied for $R = R_0 + \operatorname{diam} \operatorname{supp} \mathcal{Q}$.

2⇒1. For any bounded $\mathcal{Q} \subset \mathcal{P}$, either $\mathcal{Q} = \emptyset$ or there is a translate $\mathcal{Q}'$ of $\mathcal{Q}$ such that $0 \in \operatorname{supp} \mathcal{Q}'$. Consider the latter case. Let $R > 0$ be a constant for $\mathcal{Q}$ in condition 2. For any $a \in \mathbb{R}^d$, there is $x \in \mathbb{R}^d$ such that $\mathcal{P} \cap B(a, R) \supset \mathcal{Q}' + x$. Then $x \in \overline{B(a, R)}$ and we see $S_{\mathcal{Q}'} = \{x \in \mathbb{R}^d \mid \mathcal{Q}' + x \subset \mathcal{P}\}$ is relatively dense. Since $S_{\mathcal{Q}} = \{x \in \mathbb{R}^d \mid \mathcal{Q} + x \subset \mathcal{P}\}$ is a translate of $S_{\mathcal{Q}'}$, the set $S_{\mathcal{Q}}$ is relatively dense. □

Definition 2.39. Let $\mathcal{P}$ be a patch. A patch $\mathcal{Q}$ is $\mathcal{P}$-legal if there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}$.

Definition 2.40. Define an equivalence relation $\sim_{\mathrm{LI}}$ on $\mathrm{Patch}(\mathbb{R}^d)$ as follows. For any two patches $\mathcal{P}_1, \mathcal{P}_2$, we have $\mathcal{P}_1 \sim_{\mathrm{LI}} \mathcal{P}_2$ if and only if $\mathcal{P}_1$-legality and $\mathcal{P}_2$-legality are equivalent for bounded patches, that is,

- for any bounded $\mathcal{Q} \subset \mathcal{P}_1$ there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}_2$, and
- for any bounded $\mathcal{Q} \subset \mathcal{P}_2$ there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}_1$.

Two patches $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P}_1 \sim_{\mathrm{LI}} \mathcal{P}_2$ are said to be locally indistinguishable. The equivalence class including $\mathcal{P}$ is represented by $[\mathcal{P}]_{\mathrm{LI}}$.

Lemma 2.41. *Take $R > 0$ and $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$ arbitrarily. Then $[\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d) \subset X_{\mathcal{T}}$.*

Proof. Take $\mathcal{T}' \in [\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d)$. For any compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$, there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' \sqcap K + x \subset \mathcal{T}$. Since $\mathcal{T}' \sqcap K$ covers K, by Lemma 2.19 we have $\mathcal{T}' \cap K = (\mathcal{T} - x) \cap K$. This implies that $\mathcal{T} - x \in \mathcal{U}_{K,V}(\mathcal{T}')$. Since K and V were arbitrary, $\mathcal{T}' \in X_{\mathcal{T}}$. □

Lemma 2.42. *For any tiling $\mathcal{T}$ of $\mathbb{R}^d$, $\mathcal{S} \in X_{\mathcal{T}}$ and bounded $\mathcal{P} \subset \mathcal{S}$, there is $x \in \mathbb{R}^d$ such that $\mathcal{P} + x \subset \mathcal{T}$.*

Proof. Set $K = \operatorname{supp} \mathcal{P}$ and take an arbitrary compact neighborhood V of $0 \in \mathbb{R}^d$. There is $x \in \mathbb{R}^d$ such that $\mathcal{T} + x \in \mathcal{U}_{K,V}(\mathcal{S})$. Then there is $y \in V$ such that $\mathcal{P} = \mathcal{S} \cap K = (\mathcal{T} + x + y) \cap K$, and $\mathcal{P} - x - y \subset \mathcal{T}$. □

Proposition 2.43. *Take $R > 0$ and $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$ arbitrarily. Consider the following three conditions;*

1. *$\mathcal{T}$ is repetitive.*
2. *$[\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d) = X_{\mathcal{T}}$.*
3. *The action $\mathbb{R}^d \curvearrowright X_{\mathcal{T}}$ is minimal.*

Then always condition 1 implies condition 2 and condition 2 and condition 3 are equivalent. If $\mathcal{T}$ has FLC, then condition 2 implies condition 1.

Proof. 1⇒2. Take $\mathcal{T}' \in X_{\mathcal{T}}$. If $\mathcal{P} \subset \mathcal{T}$ is a bounded patch, there is $R_0 > 0$ such that for any $a \in \mathbb{R}^d$ there is $x \in \mathbb{R}^d$ with $\mathcal{T} \cap B(a, R_0) \supset \mathcal{P} + x$. Set $K = \overline{B(0, R_0)}$ and take an arbitrary compact neighborhood V of $0 \in \mathbb{R}^d$. There exists $x \in \mathbb{R}^d$ such that $\mathcal{T} + x \in \mathcal{U}_{K,V}(\mathcal{T}')$. This means that there is $y \in V$ such that $(\mathcal{T} + x + y) \cap K = \mathcal{T}' \cap K$. By the property of R_0 there is $z \in \mathbb{R}^d$ such that $\mathcal{T} \cap B(-x - y, R_0) \supset \mathcal{P} + z$. Then

$$\mathcal{P} + x + y + z \subset (\mathcal{T} + x + y) \cap K = \mathcal{T}' \cap K,$$

and so $\mathcal{P} + x + y + z \subset \mathcal{T}'$. By Lemma 2.42 we have $\mathcal{T}' \in [\mathcal{T}]_{\mathrm{LI}}$. Hence $X_{\mathcal{T}} \subset [\mathcal{T}]_{\mathrm{LI}}$. Since $\mathrm{Tiling}_R(\mathbb{R}^d)$ is closed with respect to the local matching topology in $\mathrm{Patch}(\mathbb{R}^d)$, $X_{\mathcal{T}} \subset \mathrm{Tiling}_R(\mathbb{R}^d)$ and together with Lemma 2.41 we obtain condition 2.

2.⇒3. Take $\mathcal{T}', \mathcal{T}'' \in X_{\mathcal{T}}$. Take a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. By condition 2 there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' \cap (K + B(0, R)) + x \subset \mathcal{T}''$, and $\mathcal{T}' \cap K = (\mathcal{T}'' - x) \cap K$ by Lemma 2.19. This means that $\mathcal{T}'' - x \in \mathcal{U}_{K,V}(\mathcal{T}')$.

3⇒2. Take $\mathcal{T}' \in X_{\mathcal{T}}$. Take an arbitrary bounded non-empty patch $\mathcal{P} \subset \mathcal{T}$. Set $K = \operatorname{supp} \mathcal{P}$ and take a compact neighborhood V of $0 \in \mathbb{R}^d$. By minimality there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' + x \in \mathcal{U}_{K,V}(\mathcal{T})$. There is $y \in V$ such that $(\mathcal{T}' + x + y) \cap K = \mathcal{T} \cap K \supset \mathcal{P}$ and $\mathcal{P} - x - y \subset \mathcal{T}'$. By Lemma 2.41 and Lemma 2.42 we obtain condition 2.

Finally we assume that $\mathcal{T}$ has FLC and satisfies condition 2 and we will prove condition 1. Suppose conversely that $\mathcal{T}$ is not repetitive. Then there are bounded $\mathcal{P} \subset \mathcal{T}$, $a_1, a_2, \ldots \in \mathbb{R}^d$ and $R_1, R_2, \ldots > 0$ such that

- The sequence (R_n) is monotone increasing and $\lim R_n = \infty$, and
- For each n the patch $\mathcal{T} \cap B(a_n, R_n)$ does not contain any translates of $\mathcal{P}$.

By Corollary 2.35 we can take a subsequence $(\mathcal{T} - a_{n_j})_j$ of the sequence $(\mathcal{T} - a_n)_n$ that converges to a tiling $\mathcal{T}_0 \in X_{\mathcal{T}}$. For any $R > 0$ and any compact neighborhood V of $0 \in \mathbb{R}^d$ there is $j_0 \in \mathbb{Z}_{>0}$ such that

$$j \geq j_0 \Rightarrow \mathcal{T} - a_{n_j} \in \mathcal{U}_{\overline{B(0,R)},V}(\mathcal{T}_0).$$

For large j, there is $x_j \in V$ such that

$$\begin{aligned}\mathcal{T}_0 \cap \overline{B(0,R)} &= (\mathcal{T} - a_{n_j} + x_j) \cap \overline{B(0,R)} \\ &\subset ((\mathcal{T} - a_{n_j}) \cap B(0, R_{n_j})) + x_j.\end{aligned}$$

This means there are no translates of $\mathcal{P}$ inside $\mathcal{T}_0 \cap B(0,R)$. Since R was arbitrary, there are no translates of $\mathcal{P}$ inside $\mathcal{T}_0$ and so $\mathcal{T}_0 \notin [\mathcal{T}]_{\mathrm{LI}}$. This contradicts condition 2. $\square$

§ 2.4. A result on relation between properties of tilings and properties of the corresponding dynamical systems

Proposition 2.43 describes a relation between distribution of patches in a tiling and a property of the corresponding dynamical system. Here we mention another relation.

The definitions for eigenvalues and eigenfunctions are given in the appendix. For $\xi \in \mathbb{R}^d$, the character of $\mathbb{R}^d$, $\mathbb{R}^d \ni x \mapsto e^{2\pi i \langle x, \xi \rangle}$, is denoted by χ_ξ. ($\langle (x_i), (y_i) \rangle = \sum_i x_i y_i$ is the standard inner product.)

Theorem 2.44 ([13]). *Let $\mathcal{T}$ be a repetitive tiling in $\mathbb{R}^d$ of FLC and FTT. Let A be a subgroup of the group of all topological eigenvalues for the topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. Assume $0 \in \overline{A \setminus \{0\}}$. Then for any $R_0 > 0$ and $\varepsilon > 0$, there exist $R > 0$ and $\xi \in A \setminus \{0\}$ that satisfy the following two conditions:*

- *for any $\mathcal{T}$-legal finite patches $\mathcal{P}_1, \mathcal{P}_2$ such that each of $\operatorname{supp} \mathcal{P}_1$ and $\operatorname{supp} \mathcal{P}_2$ contains a ball of radius R, the set*

  $$\{x \in \mathbb{R}^d \mid \mathcal{P}_1 \cup (\mathcal{P}_2 + x) \text{ is not } \mathcal{T}\text{-legal}\}$$

  contains a translate of $B(0, R_0) + \operatorname{Ker} \chi_\xi$.

- $$8R_0 < \frac{1}{\|\xi\|} < (8 + \varepsilon) R_0. \tag{2.6}$$

Remark. This theorem says that, given the property of the group of topological eigenvalues, there is a "forbidden area" of appearance of translates of $\mathcal{P}_2$ in $\mathcal{T}$ relative to any translate of $\mathcal{P}_1$ in $\mathcal{T}$. The forbidden area is belt-like and is obtained by juxtaposing translates of

$B(0, R_0) + \{\xi\}^{\perp}$ ($^{\perp}$ means orthogonal complement) with interval $\frac{1}{\|\xi\|}$, which is approximately $8R_0$.

By Theorem 3.37, for tilings from a substitution rule σ that satisfies the conditions of Theorem 3.37, if the spectrum of the expansive map forms a Pisot family, the group of eigenvalues is relatively dense. If moreover the map $\omega_\sigma \colon X_\sigma \to X_\sigma$ is injective, then the group of eigenvalues is dense because if ξ is an eigenvalue, so is $(\varphi^*)^{-1}(\xi)$ by Theorem 3.35 and non-periodicity of tilings in X_σ (Theorem 3.33). Thus under such conditions, we can apply Theorem 2.44 to the tilings $\mathcal{T} \in X_\sigma$.

§ 3. Substitution rules

As was mentioned there are several ways to construct tilings of $\mathbb{R}^d$. In this section we introduce one of the ways, namely the way from substitution rules. After definitions we introduce some of important results.

Definition 3.1. Let $\mathcal{A}$ be a finite set of tiles in $\mathbb{R}^d$. Set

$$\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) = \{\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d) \mid \text{any tile } T \in \mathcal{P} \text{ is a translate of a tile in } \mathcal{A}\}.$$

Lemma 3.2. *The set* $\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ *is a closed subset of* $\mathrm{Patch}(\mathbb{R}^d)$ *with respect to the local matching topology.*

Proof. Take $\mathcal{P} \in \overline{\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)}$ and $T \in \mathcal{P}$. $\mathcal{P}$ and an element $\mathcal{Q} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ coincide, after a small translation, inside a large ball around the origin. Thus for some $\mathcal{Q} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ a translate of T appears in $\mathcal{Q}$ and T is a translate of an element of $\mathcal{A}$. $\square$

Definition 3.3. A linear map $\varphi \colon \mathbb{R}^d \to \mathbb{R}^d$ is said to be expansive if

- it is bijective, and
- if λ is any eigenvalue for φ, then $|\lambda| > 1$.

Definition 3.4. A substitution rule (of $\mathbb{R}^d$) is a triple $(\mathcal{A}, \varphi, \omega)$ where

- $\mathcal{A}$ is a finite nonempty set of tiles in $\mathbb{R}^d$,
- φ is an expansive linear map of $\mathbb{R}^d$, and
- ω is a map $\omega \colon \mathcal{A} \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ such that

$$\operatorname{supp} \omega(P) = \overline{\varphi(P)}.$$

Tiles in $\mathcal{A}$ are called proto-tiles for the substitution rule.

Remark. Roughly speaking, a substitution rule is a way to expand each proto-tile, subdivide it and obtain a patch consisting of translates of proto-tiles. The following example will illuminate this point.

We can also consider substitution rules with rotation or flip. Radin's pinwheel tiling [15] is an example. We do not deal with such substitution rules in this article.

Example 3.5 (Figure1). Set $\tau = \frac{1+\sqrt{5}}{2}$. Take the interior of the triangle which has side-length 1,1, and τ, and remove one point anywhere from the left side or the right side. Moreover take the interior of the triangle of the side-length τ, τ and 1, and remove one point from the left side or the right side. The proto-tiles of this substitution are the copies of these two punctured triangles by $2n\pi/10$-rotations and flip, where $n = 0, 1, \dots, 9$. There are 40 proto-tiles.

The expansion map is τI, where I is the identity map. The map ω is depicted in Figure 1. The image of the other proto-tiles by ω is defined accordingly, so that ω and rotation, ω and flip will commute.

Tilings for this substitution are called Robinson triangle tilings. Such tilings are known to be related (MLD) to Penrose tilings by kite and dart.

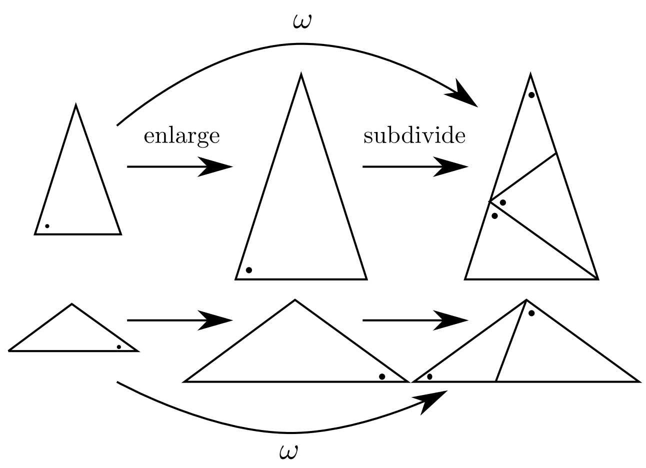

Figure 1. Example of substitution

Definition 3.6. For a substitution rule $(\mathcal{A}, \varphi, \omega)$, $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$, we set a patch $\omega(P + x) \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ by

$$\omega(P + x) = \omega(P) + \varphi(x).$$

An easy computation shows the next lemma:

Lemma 3.7. *Let $(\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then* $\operatorname{supp} \omega(P + x) = \varphi(\overline{P}) + \varphi(x) = \varphi(\overline{P + x})$.

Definition 3.8. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Define a map

$$\omega_\sigma \colon \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$$

by

$$\omega_\sigma(\mathcal{P}) = \bigcup_{T \in \mathcal{P}} \omega(T).$$

Lemma 3.9. *For any substitution rule σ, the map ω_σ is well defined, that is, for any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ we have $\omega_\sigma(\mathcal{P}) \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Moreover the following conditions hold:*

- *For any* $\mathcal{P}_1, \mathcal{P}_2, \dots \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$*, if* $\bigcup_n \mathcal{P}_n$ *is a patch, then we have* $\omega_\sigma(\bigcup \mathcal{P}_n) = \bigcup \omega_\sigma(\mathcal{P}_n)$*.*
- *For any* $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$*,* $\operatorname{supp} \omega_\sigma(\mathcal{P}) = \varphi(\operatorname{supp} \mathcal{P})$*.*
- *For any* $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$*,* $x \in \mathbb{R}^d$ *and* $m \in \mathbb{Z}_{>0}$*,* $\omega_\sigma^m(\mathcal{P} + x) = \omega_\sigma^m(\mathcal{P}) + \varphi^m(x)$*.*

The following lemma also holds for the local matching topology, but we omit the proof.

Lemma 3.10. *For any substitution rule* $\sigma = (\mathcal{A}, \varphi, \omega)$*, the map* ω_σ *is continuous with respect to the cylinder topology.*

Proof. Take $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Take any finite $\mathcal{Q} \subset \omega_\sigma(\mathcal{P})$ and a neighborhood U of 0 in $\mathbb{R}^d$ (cf. Lemma 2.5). For any $T \in \mathcal{Q}$ there is $S_T \in \mathcal{P}$ such that $T \in \omega(S_T)$. Set $\mathcal{P}' = \{S_T \mid T \in \mathcal{Q}\}$ and $U' = \varphi^{-1}(U)$. Then $\omega_\sigma(C(U', \mathcal{P}') \cap \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)) \subset C(U, \mathcal{Q})$. $\square$

Remark. Often in the literature the letter σ is suppressed and ω_σ is simply written as ω. Of course, $\omega_\sigma(\{P + x\}) = \omega(P + x)$ for $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$.

Definition 3.11. A substitution rule $(\mathcal{A}, \varphi, \omega)$ is said to be primitive if the following condition holds:

$$\text{There is } K \in \mathbb{Z}_{>0} \text{ such that, if } P \text{ and } P' \text{ are in } \mathcal{A}, \text{ then there is } x \in \mathbb{R}^d \text{ such that } P + x \in \omega_\sigma^K(\{P'\}).$$

Definition 3.12. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. A patch $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ is said to be σ-legal if there are $P \in \mathcal{A}, n \in \mathbb{Z}_{>0}$ and $x \in \mathbb{R}^d$ such that

$$\mathcal{P} \subset \omega_\sigma^n(\{P + x\}).$$

Definition 3.13. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule of $\mathbb{R}^d$. Define

$$X_\sigma = \{\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d) \mid \text{if } \mathcal{P} \subset \mathcal{T} \text{ is a finite patch, then } \mathcal{P} \text{ is } \sigma\text{-legal}\}.$$

In the following arguments we show X_σ is not empty.

Lemma 3.14. *Let* $\sigma = (\mathcal{A}, \varphi, \omega)$ *be a substitution rule. There are* $P \in \mathcal{A}$*,* $m > 0$*, and* $x \in \mathbb{R}^d$ *such that*

- $P + x \in \omega_\sigma^m(\{P\})$*, and*
- $\overline{P + x} \subset \varphi^m(P)$*.*

Proof. Since φ is expansive, for any $P \in \mathcal{A}$, there are $m > 0, x \in \mathbb{R}^d$ and $P' \in \mathcal{A}$ such that

$$P' + x \in \omega_\sigma^m(\{P\}), \text{ and} \tag{3.1}$$

$$\overline{P' + x} \subset \varphi^m(P). \tag{3.2}$$

If for some m, x the conditions (3.1) and (3.2) hold, we write $P \rightsquigarrow P'$.

We have a sequence $P_1, P_2, \dots$ of $\mathcal{A}$ such that for each n we have $P_n \rightsquigarrow P_{n+1}$. Since $\mathcal{A}$ is finite, for some k, l with $k < l$ we obtain $P_k = P_l$. Thus it suffices to show that if P, P', P'' are in $\mathcal{A}$ and $P \rightsquigarrow P'$ and $P' \rightsquigarrow P''$ hold, then $P \rightsquigarrow P''$. But this is clear by a simple computation. $\square$

Lemma 3.15. *Take a finite nonempty set $\mathcal{A}$ of tiles. Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map. Let $\omega\colon \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ be a map such that $\operatorname{supp}\omega(\mathcal{P}) = \varphi(\operatorname{supp}\mathcal{P})$ for each $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Suppose there is $\mathcal{P}_0 \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ such that*

- $\mathcal{P}_0 \subset \omega(\mathcal{P}_0)$, *and*
- $\operatorname{supp}\mathcal{P}_0 \subset \varphi(\operatorname{supp}\mathcal{P}_0)^\circ$.

Then there is $r > 0$ such that $\operatorname{supp}\omega^n(\mathcal{P}_0) \supset \varphi^n(B(0,r))$ for any $n \in \mathbb{Z}_{>0}$.

Proof. For each n we have $(\operatorname{supp}\mathcal{P}_0)^\circ \supset \varphi^{-n}(\operatorname{supp}\mathcal{P}_0) \supset \varphi^{-n-1}(\operatorname{supp}\mathcal{P}_0)$. Take $x \in \varphi^{-1}(\operatorname{supp}\mathcal{P}_0)$. Then $0 = \lim_n \varphi^{-n}(x) \in \varphi^{-1}(\operatorname{supp}\mathcal{P}_0) \subset (\operatorname{supp}\mathcal{P}_0)^\circ$. There exists $r > 0$ such that $B(0,r) \subset \operatorname{supp}\mathcal{P}_0$. For each n

$$\begin{aligned}\operatorname{supp}\omega^n(\mathcal{P}_0) &= \varphi(\operatorname{supp}\omega^{n-1}\mathcal{P}_0)\\ &= \varphi^2(\operatorname{supp}\omega^{n-2}\mathcal{P}_0)\\ &= \cdots\\ &= \varphi^n(\operatorname{supp}\mathcal{P}_0)\\ &\supset \varphi^n(B(0,r)).\end{aligned}$$

□

Proposition 3.16. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then there are $P \in \mathcal{A}$, $b \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$ such that $P + b \in \omega_\sigma^m(\{P+b\})$ and*

$$\bigcup_{n>0} \omega_\sigma^{nm}(\{P+b\})$$

is a tiling in X_σ.

Proof. By Lemma 3.14, there are $P \in \mathcal{A}, a \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned}P + a &\in \omega_\sigma^m(\{P\}), \text{ and}\\ \overline{P+a} &\subset \varphi^m(P).\end{aligned}$$

Since φ is expansive, a linear map $I - \varphi^m$ is invertible. Set $b = (I - \varphi^m)^{-1}(a)$. Then we have

$$P + b \in \omega_\sigma^m(\{P+b\}), \text{ and} \tag{3.3}$$

$$\overline{P+b} \subset \varphi^m(P+b). \tag{3.4}$$

Set

$$\mathcal{T} = \bigcup_{n>0} \omega_\sigma^{nm}(\{P+b\}).$$

By (3.3), $\mathcal{T}$ is a patch. Moreover $\operatorname{supp}\omega_\sigma^m(\mathcal{P}) = \varphi^m(\operatorname{supp}\mathcal{P})$ for any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ and $\operatorname{supp}\{P+b\} \subset \varphi^m((\operatorname{supp}\{P+b\})^\circ)$. Applying Lemma 3.15 for $\mathcal{P}_0 = \{P+b\}$ and $\omega = \omega_\sigma^m$, we see $\operatorname{supp}\mathcal{T} = \mathbb{R}^d$ and so $\mathcal{T}$ is a tiling.

Finally if $\mathcal{P}$ is a finite subset of $\mathcal{T}$, then for some n, the patch $\mathcal{P}$ is included in $\omega_\sigma^{mn}(\{P+b\})$ and so $\mathcal{P}$ is σ-legal. Thus $\mathcal{T}$ is in X_σ. □

Lemma 3.17. *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ such that $\sup_{T\in\mathcal{T}} \operatorname{diam} T < r$ for some $r > 0$. Then for any subset $S \subset \mathbb{R}^d$, $\operatorname{supp}\mathcal{T} \cap (S + \overline{B(0,r)}) \supset S$.*

Lemma 3.18. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then X_σ is closed in* $\operatorname{Patch}(\mathbb{R}^d)$ *(and in* $\operatorname{Patch}_{\mathcal{A}}(\mathbb{R}^d)$*) with respect to the local matching topology.*

Proof. Take $\mathcal{T} \in \overline{X_\sigma}$. For any compact $K \subset \mathbb{R}^d$ and any compact neighborhood V of $0 \in \mathbb{R}^d$, there is $\mathcal{T}' \in \mathcal{U}_{K,V}(\mathcal{T}) \cap X_\sigma$. We can take $x \in V$ such that $\mathcal{T} \cap K = (\mathcal{T}' + x) \cap K$. Thus if $\mathcal{P}$ is a finite subset of $\mathcal{T}$, by taking K large enough, we see that there are $\mathcal{T}' \in X_\sigma$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} - x \subset \mathcal{T}'$. Since $\mathcal{P} - x$ is σ-legal, $\mathcal{P}$ is also σ-legal. Next, for any compact L set $K = L + \overline{B(0,r)}$ where $r > \max_{P\in\mathcal{A}} \operatorname{diam} P$. By the above argument and Lemma 3.17

$$\begin{aligned} \operatorname{supp}\mathcal{T} &\supset \operatorname{supp}\mathcal{T} \cap K \\ &= \operatorname{supp}(\mathcal{T}' + x) \cap K \\ &\supset L. \end{aligned}$$

for some $\mathcal{T}' \in X_\sigma$ and $x \in \mathbb{R}^d$. It follows that $\operatorname{supp}\mathcal{T} = \mathbb{R}^d$ and so $\mathcal{T} \in X_\sigma$. □

Remark. By Proposition 2.21, X_σ is closed in $\operatorname{Tiling}_R(\mathbb{R}^d)$ with respect to the cylinder topology for any $R > \max_{P\in\mathcal{A}} \operatorname{diam} P$.

Remark. Since a translate of σ-legal patch is again σ-legal, it is clear that X_σ is invariant under translation.

Lemma 3.19. *If $\mathcal{T} \in X_\sigma$, then $\omega_\sigma(\mathcal{T}) \in X_\sigma$.*

Proof. Take a finite $\mathcal{P} \subset \omega_\sigma(\mathcal{T})$. For any $T \in \mathcal{P}$ there is $S_T \in \mathcal{T}$ such that $T \in \omega(S_T)$. Set $\mathcal{P}' = \{S_T \mid T \in \mathcal{P}\}$, then $\mathcal{P} \subset \omega_\sigma(\mathcal{P}')$. Since $\mathcal{P}'$ is σ-legal, there are P, x, n such that $\mathcal{P}' \subset \omega_\sigma^n(\{P+x\})$, and $\mathcal{P} \subset \omega_\sigma^{n+1}(\{P+x\})$. This means that $\mathcal{P}$ is σ-legal. Moreover $\operatorname{supp}\omega_\sigma(\mathcal{T}) = \varphi(\operatorname{supp}\mathcal{T}) = \mathbb{R}^d$ by Lemma 3.9. □

Proposition 3.20 ([1], Proposition 2.2)**.** *Let $(\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then $\omega_\sigma\colon X_\sigma \to X_\sigma$ is surjective.*

The following easy lemmas will be useful later.

Lemma 3.21. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule and take $n \in \mathbb{Z}_{>0}$. Then*

$$\omega_\sigma^n(\mathcal{P}) = \bigcup_{T\in\mathcal{P}} \omega_\sigma^n(\{T\})$$

for any $\mathcal{P} \in$ $\operatorname{Patch}_{\mathcal{A}}(\mathbb{R}^d)$.

Definition 3.22. For a substitution rule $\sigma = (\mathcal{A}, \varphi, \omega)$ and $n \in \mathbb{Z}_{>0}$, define a substitution rule σ^n by $\sigma^n = (\mathcal{A}, \varphi^n, \omega^n)$ where $\omega^n(P) = \omega_\sigma^n(\{P\})$ for each $P \in \mathcal{A}$.

Remark. If σ is primitive, then so is σ^n for any n.

Lemma 3.23. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule and take $n \in \mathbb{Z}_{>0}$. Then $(\omega_\sigma)^n = (\omega^n)_{\sigma^n}$ (the iterate of ω_σ coincides with the map associated to σ^n in regard to Definition 3.8).*

Lemma 3.24. *Let σ be a primitive substitution. Then for any $n \in \mathbb{Z}_{>0}$ we have $X_\sigma = X_{\sigma^n}$.*

Proof. Take $\mathcal{T} \in X_{\sigma^n}$ and a finite subset $\mathcal{P} \subset \mathcal{T}$. There are $P \in \mathcal{A}, m > 0$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^{nm}(\{P + x\})$ (cf. Lemma 3.23). This shows that $\mathcal{P}$ is σ-legal and $\mathcal{T} \in X_\sigma$.

Next, take $\mathcal{T} \in X_\sigma$ and finite $\mathcal{P} \subset \mathcal{T}$. There are $P \in \mathcal{A}, m > 0$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^m(\{P + x\})$. There is $K \in \mathbb{Z}_{>0}$ as in Definition 3.11. Take $l \in \mathbb{Z}_{>0}$ such that $nl \geq K + m$. We can take $y \in \mathbb{R}^d$ such that $P + y \in \omega_\sigma^{nl-m}(\{P\})$. Then

$$\mathcal{P} \subset (\omega_{\sigma^n}^n)^l(\{P + \varphi^{m-nl}(x - y)\}),$$

and so $\mathcal{P}$ is σ^n-legal. □

Definition 3.25. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. If the set

$$\{\omega_\sigma^n(\{P\}) \cap B(x, R) \mid P \in \mathcal{A}, n > 0, x \in \mathbb{R}^d\}/\sim$$

is finite for each $R > 0$, then σ is said to have FLC.

Note that by Proposition 2.21, on X_σ the relative topologies of the local matching topology and the cylinder topology coincide. We endow X_σ this relative topology.

Lemma 3.26. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a primitive substitution rule. Then the following conditions are equivalent:*

1. *σ has FLC.*
2. *X_σ has FLC.*
3. *X_σ is compact.*

Proof. 1⇒2. Suppose σ has FLC. Take a positive number $R > 0$. Take $\mathcal{T} \in X_\sigma$ and $x \in \mathbb{R}^d$, and set $\mathcal{P} = \mathcal{T} \cap B(x, R)$. By definition of X_σ there are $P \in \mathcal{A}, n > 0$ and $y \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^n(\{P + y\})$. For some $z \in \mathbb{R}^d$ a translate of $\mathcal{P}$ appears inside $\omega_\sigma^n(\{P\}) \cap B(z, R)$. Thus by Lemma 2.27,

$$\{\mathcal{T} \cap B(x, R) \mid x \in \mathbb{R}^d, \mathcal{T} \in X_\sigma\}/\sim$$

is finite.

2⇒1. Take $R > 0$ arbitrarily. If $P \in \mathcal{A}$, then by primitivity and Lemma 3.19, there is $\mathcal{T} \in X_\sigma$ such that $P \in \mathcal{T}$. Take $n \in \mathbb{Z}_{>0}$ and $x \in \mathbb{R}^d$. Then $\omega_\sigma^n(\{P\}) \cap B(x, R) \subset \omega_\sigma^n(\mathcal{T}) \cap B(x, R)$. Since $\omega_\sigma^n(\mathcal{T}) \in X_\sigma$ (Lemma 3.19), by Lemma 2.27 and condition 2,

$$\{\omega_\sigma^n(\{P\}) \cap B(x, R) \mid P \in \mathcal{A}, n > 0, x \in \mathbb{R}^d\}/\sim$$

is finite.

The equivalence of 2 and 3 follows from Lemma 3.18 and Lemma 2.34. □

Remark. It is known that given a substitution rule $\sigma = (\mathcal{A}, \varphi, \omega)$, it is often possible to prove FLC of σ by observing "coronas" in iterates $\omega_\sigma^n(\{P\})$ for any $P \in \mathcal{A}$ and small $n \in \mathbb{Z}_{>0}$.

Remark. By Lemma 3.24 and Lemma 3.26, we see that for any primitive σ and $n > 0$, σ has FLC if and only if σ^n has FLC.

If σ has FLC we obtain a topological dynamical system $(X_\sigma, \mathbb{R}^d)$ by action by translation.

Proposition 3.27. *If $\sigma = (\mathcal{A}, \varphi, \omega)$ is primitive, then $(X_\sigma, \mathbb{R}^d)$ is minimal and any $\mathcal{T} \in X_\sigma$ is repetitive.*

Proof. Let K be a positive integer appeared in Definition 3.11. Take a real number $r > \max_{P \in \mathcal{A}} \operatorname{diam} P$. Take $\mathcal{T}, \mathcal{S} \in X_\sigma$ and a finite $\mathcal{P} \subset \mathcal{T}$ arbitrarily. By definition of X_σ, there are $P \in \mathcal{A}, y \in \mathbb{R}^d$ and $n > 0$ such that $\mathcal{P} \subset \omega_\sigma^n(\{P + y\})$. Take $R > 0$ such that $\varphi^{K+n} B(0, r) \subset B(0, R)$. We claim

$$\text{for any } x \in \mathbb{R}^d \text{, there is a translate of } \mathcal{P} \text{ in } \mathcal{S} \cap B(x, R). \tag{3.5}$$

Take $x \in \mathbb{R}^d$. By Proposition 3.20, there is $\mathcal{S}' \in X_\sigma$ such that $\omega_\sigma^{n+K}(\mathcal{S}') = \mathcal{S}$. We can take $T \in \mathcal{S}'$ such that $\varphi^{-n-K}(x) \in \overline{T}$. Then there is a translate of P in $\omega_\sigma^K(\{T\})$, and there is a translate of $\mathcal{P}$ in $\omega_\sigma^{n+K}(\{T\})$. Since $\mathcal{S} \supset \omega_\sigma^{n+K}(\{T\})$ and $\operatorname{supp} \omega_\sigma^{n+K}(\{T\}) \subset B(x, R)$, there is a translate of $\mathcal{P}$ in $\mathcal{S} \cap B(x, R)$. Thus the clain (3.5) is proved. This firstly means that a translate of $\mathcal{S}$ contains $\mathcal{P}$. By Lemma 2.5, this implies that for any neighborhood of $\mathcal{T}$, a translate of $\mathcal{S}$ is a member of that neighborhood. This means that $(X_\sigma, \mathbb{R}^d)$ is minimal. Secondly the claim (3.5) shows that (by considering the case where $\mathcal{S} = \mathcal{T}$) $\mathcal{T}$ is repetitive. □

Remark. This proposition shows that, if σ is primitive then $X_\sigma = X_{\mathcal{S}}$ for any $\mathcal{S} \in X_\sigma$.

Definition 3.28. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. A tiling $\mathcal{T} \in \operatorname{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ is called a fixed point if $\omega_\sigma(\mathcal{T}) = \mathcal{T}$. A repetitive tiling of FLC which is a fixed point of some substitution rule is called a self-affine tiling.

Lemma 3.29. *Let σ be a substitution rule and $\mathcal{T}$ be its fixed point. If $\mathcal{T}$ is repetitive, then $\mathcal{T} \in X_\sigma$. If σ is primitive and $\mathcal{T} \in X_\sigma$, then $\mathcal{T}$ is repetitive.*

Proof. Suppose $\mathcal{T}$ is repetitive. Take a finite $\mathcal{P} \subset \mathcal{T}$. There exists $R > 0$ such that for any $x \in \mathbb{R}^d$ there is $y \in \mathbb{R}^d$ with $\mathcal{T} \cap B(x, R) \supset \mathcal{P} + y$. For arbitrary $T \in \mathcal{T}$, if n is large enough the support of the patch $\omega_\sigma^n(\{T\})$ contains a ball of radius R. Hence a translate of $\mathcal{P}$ appears in $\omega_\sigma^n(\{T\})$, and so $\mathcal{P}$ is σ-legal. Hence $\mathcal{T} \in X_\sigma$. The converse under the assumption of primitivity is proved in Proposition 3.27. □

Lemma 3.30. *For any primitive substitution rule σ there is $n > 0$ such that σ^n admits a repetitive fixed point.*

Proof. This is clear by Proposition 3.16, Lemma 3.24 and Lemma 3.29. □

Remark. Often we assume a primitive substitution admits a repetitive fixed point because we may replace the original substitution σ with σ^n for some n.

Theorem 3.31 ([23], [10])**.** *If a substitution rule σ is primitive and FLC, then the corresponding topological dynamical system $(X_\sigma, \mathbb{R}^d)$ is uniquely ergodic, that is, it admits a unique invariant probability measure.*

We recall mixing property of dynamical systems in Definition 4.7.

Theorem 3.32 ([23], Theorem 4.1)**.** *Let σ be a primitive substitution of FLC. Then the dynamical system $(X_\sigma, \mathbb{R}^d, \mu)$ is not mixing, where μ is the unique invariant probability measure.*

Remark. The proof of the previous theorem is decomposed into two parts. Let $\mathcal{T}$ be a repetitive fixed point. First, we can prove the following: take any $T \in \mathcal{T}$ and any vector x such that $T + x \in \mathcal{T}$. Then there is $c > 0$ such that for any finite patch $\mathcal{P}$ and $n \in \mathbb{Z}_{>0}$, we have

$$\lim_{N} \frac{L(\mathcal{P} \cup (\mathcal{P} + \varphi^n(x)), \mathcal{T} \cap A_N)}{m(A_N)} \geqq c \frac{L(\mathcal{P}, \omega^n(T))}{m(\varphi^n(T))}. \tag{3.6}$$

Here,

- $L(\mathcal{P}, \mathcal{Q}) = \operatorname{card}\{x \in \mathbb{R}^d \mid \mathcal{P} + x \subset \mathcal{Q}\}$ (the number of translates of $\mathcal{P}$ inside $\mathcal{Q}$) for any patch $\mathcal{P}, \mathcal{Q}$.
- A_N is the ball of radius N with its center 0, or more generally (A_N) is a van Hove sequence.

The left-hand side of inequality (3.6) is called the frequency of the patch $\mathcal{P} \cup (\mathcal{P} + \varphi^n(x))$. Roughly speaking, this inequality means that the probability of finding another translate of $\mathcal{P}$ if we find a translate of $\mathcal{P}$ in the tiling $\mathcal{T}$ and move our attention by a vector $\varphi^n(x)$ from that position, is bounded below.

Next, from this fact about distribution of patches we can prove the property of the dynamical system, i.e. that the dynamical system is not mixing. This is an example of a relation between distribution of patches in tilings and corresponding tiling dynamical systems.

Solomyak [22] proved recognizability of certain substitution rules, which is a tiling analogue of [12].

Theorem 3.33 ([22])**.** *Let σ be a primitive substitution rule of FLC. Then the map $\omega_\sigma \colon X_\sigma \to X_\sigma$ is injective if and only if each $\mathcal{T} \in X_\sigma$ is non-periodic.*

In this theorem the "if" part is difficult. For the "only if" part see for example [1], Proposition 2.3.

For examples of substitution rule the following lemma is useful to prove that ω_σ is injective.

Lemma 3.34. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Suppose that the following three conditions*

- *$P \in \mathcal{A}$,*

- *$\mathcal{P}$ is a σ-legal finite patch, and*
- *$\omega(P) \subset \omega_\sigma(\mathcal{P})$,*

imply $P \in \mathcal{P}$. Then $\omega_\sigma \colon X_\sigma \to X_\sigma$ is injective.

Proof. Take $\mathcal{T}, \mathcal{S} \in X_\sigma$ and assume $\omega_\sigma(\mathcal{T}) = \omega_\sigma(\mathcal{S})$. Take $T \in \mathcal{T}$ arbitrarily. Set $\mathcal{P} = \mathcal{S} \sqcap \overline{T}$. Then $\operatorname{supp} \omega(T) \subset \operatorname{supp} \omega_\sigma(\mathcal{P})$. Since $\omega(T) \subset \omega_\sigma(\mathcal{S})$, we have $\omega(T) \subset \omega_\sigma(\mathcal{P})$. There are $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$ such that $T = P + x$. We have $\omega(P) \subset \omega_\sigma(\mathcal{P} - x)$ and by the assumption of this lemma we obtain $P \in \mathcal{P} - x$, and so $T \in \mathcal{P} \subset \mathcal{S}$. Hence $\mathcal{T} \subset \mathcal{S}$ and so $\mathcal{T} = \mathcal{S}$. $\square$

By the following theorem we see for certain dynamical systems from substitution, topological and measurable eigenvalues coincide and any measurable eigenfunction can be taken continuous. (These notions are explained in Appendix.)

Theorem 3.35 ([24], Theorem 3.13)**.** *Let $(\mathcal{A}, \varphi, \omega)$ be a primitive tiling substitution of FLC. Assume there is a repetitive fixed point $\mathcal{T}$ for this substitution. Then for $\xi \in \mathbb{R}^d$, the following conditions are equivalent:*

1. *ξ is a topological eigenvalue for the topological dynamical system $(X_\sigma, \mathbb{R}^d)$;*
2. *ξ is a measurable eigenvalue for the measure-preserving system $(X_\sigma, \mathbb{R}^d, \mu)$, where μ is the unique invariant probability measure;*
3. *ξ satisfies the following two conditions:*
   (a) *For any return vector z (cf. Definition 2.2) for $\mathcal{T}$, we have*

$$\lim_{n \to \infty} e^{2\pi i \langle \varphi^n(z), \xi \rangle} = 1, \tag{3.7}$$

   and

   (b) *if $z \in \mathbb{R}^d$ and $\mathcal{T} + z = \mathcal{T}$, then*

$$e^{2\pi i \langle z, \xi \rangle} = 1.$$

Definition 3.36.

- An algebraic integer $\lambda > 1$ is called a Pisot number if all its Galois conjugates μ except λ itself satisfy $|\mu| < 1$.
- Let Λ be a finite non-empty set of algebraic integers. We say Λ is a Pisot family if the following condition holds:

$$\text{if } \lambda \in \Lambda,\ \mu \notin \Lambda \text{ and } \lambda \text{ and } \mu \text{ are Galois conjugate, then } |\mu| < 1.$$

For example, $\tau = \frac{1+\sqrt{5}}{2}$ is a Pisot number because τ and $\frac{1-\sqrt{5}}{2}$ are all of its Galois conjugates. A one-point set $\{\tau\}$ forms a Pisot family.

For a linear map $\varphi \colon \mathbb{R}^d \to \mathbb{R}^d$, its spectrum $\operatorname{sp}(\varphi)$ is by definition the set of all eigenvalues.

Theorem 3.37 ([11], Theorem 2.8). *Let $(\mathcal{A}, \varphi, \omega)$ be a primitive substitution rule of FLC admitting a repetitive fixed point. Assume φ is diagonalizable over $\mathbb{C}$ and all the eigenvalues are Galois conjugates of the same multiplicity. Then the following two conditions are equivalent:*

1. *The set* $\mathrm{sp}(\varphi)$ *is a Pisot family.*
2. *The set of (topological and measurable) eigenvalues for $(X_\sigma, \mathbb{R}^d)$ is relatively dense.*

§ 4. Appendix: generalities of dynamical systems

Definition 4.1. If X is a compact space, G a locally compact abelian group and $\alpha\colon G \curvearrowright X$ is a continuous action, then the triple (X, G, α) (or simply the pair (X, G)) is called a topological dynamical system.

We often suppress α and simply write the image of $x \in X$ by $g \in G$ by $g \cdot x$. Recall a character of G is a homomorphism $\chi\colon G \to \mathbb{T}$ where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 4.2. Let (X, G) be a topological dynamical system. A non-zero continuous function $f\colon X \to \mathbb{C}$ is called a topological eigenfunction if there is a continuous character $\chi\colon G \to \mathbb{T}$ such that $f(g \cdot x) = \chi(g) f(x)$ for any $g \in G$ and $x \in X$. The character χ is called the eigenvalue for the eigenfunction f.

Remark. A non-zero constant function is always a topological eigenfunction.

Definition 4.3. A topological dynamical system (X, G) is said to be weakly mixing if it admits no topological eigenfunctions other than constants.

Definition 4.4. A measure-preserving system is a quintuplet $(X, \mathcal{F}, \mu, G, \alpha)$ where $(X, \mathcal{F}, \mu)$ is a probability space, G a locally compact abelian group and $\alpha\colon G \curvearrowright X$ is a measure-preserving action, that is, for each $g \in G$ the map $\alpha_g\colon X \to X$ preserves measurability and measure.

Definition 4.5. Let $(X, \mathcal{F}, \mu, G, \alpha)$ be a measure-preserving system. An element $f \in \mathcal{L}^2(\mu) \setminus \{0\}$ is called a measurable eigenfunction if there is a continuous character χ such that two functions $x \mapsto f(g \cdot x)$ and $x \mapsto \chi(g) f(x)$ coincide almost everywhere for any $g \in G$. The character χ is called the eigenvalue for the eigenfunction f.

Definition 4.6. A measure-preserving system $(X, \mathcal{F}, \mu, G, \alpha)$ is said to be weakly mixing if there are no measurable eigenfunctions other than constants.

Remark. In both topological and measurable cases, if $G = \mathbb{R}^d$, we identify $\hat{\mathbb{R}^d}$ and $\mathbb{R}^d$ and say $\xi \in \mathbb{R}^d$ is an eigenvalue if the character $x \mapsto e^{2\pi i \langle \xi, x \rangle}$ is an eigenvalue for some eigenfunction.

We say a sequence $g_1, g_2, \ldots$ of G converges to infinity if for any compact $K \subset G$, we have $g_n \notin K$ eventually.

Definition 4.7. Let $(X, \mathcal{F}, \mu, G, \alpha)$ be a measure-preserving system. We say the system is mixing if whenever we take $E, F \in \mathcal{F}$ and a sequence $g_1, g_2, \ldots$ in G that converges to infinity, we have $\mu(E \cap (g_n \cdot F)) \to \mu(E)\mu(F)$.

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